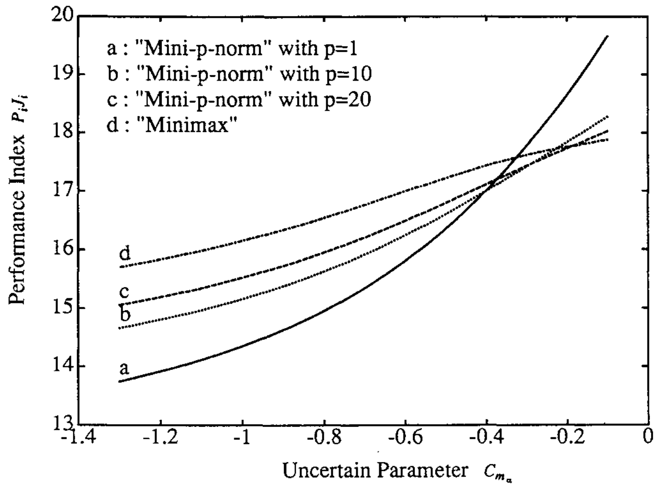

Fig. 1 α-β **estimator results for** C_{m_α} **variation of T-38.**

chosen to represent a moderately extensive pattern of jinking maneuvers. The noise intensity in the measurement of h was assumed to be 2.4 ft^2.

The C_{m_α} of the T-38 was changed from -1.3 to -0.1, which is the typical range of C_{m_α} of military aircraft. C_{m_α} is a term in the A matrix as specified in Ref. 11. Seven values of C_{m_α} equally spaced in this range were chosen as the quantization points ($k=7$), and the probability P_i was assumed to be the same for all cases of C_{m_α}.

The error variances of the estimator were calculated analytically by solving the Lyapunov equations. In Fig. 1, a–c are the results of the mini-p-norm method with $p=1$, 10, and 20, respectively, and d is the minimax estimator result, obtained from the Kalman filter designed at the worst flight condition (i.e., at $C_{m_\alpha}=-0.1$).

The sum of error variances shown as case a in Fig. 1 is the minimum we can achieve since the mini-p-norm method with $p=1$ is the same as the minisum method. As p increases, the maximum error variance decreases and the sensitivity to parameter variations is reduced, but at the cost of a larger sum of error variances. As p becomes too large, the maximum error variance decreases only very insignificant amounts, whereas the sum of error variances increases by substantial amounts. The mini-p-norm method with $p=10$ can be proposed as the most favorable design method in this tracking problem if low sensitivity and small error variance are both required. This illustrates the use of p as a design parameter in designing robust estimators.

VI. Conclusions

The mini-p-norm method for the design of estimators that are robust to parameter variations was presented. This method assumed that the estimator system matrices are fixed by the designer and finds the estimator gains minimizing the new cost criterion, which is directly related to the p norm of the performance index vector. This method has a new design parameter with which a tradeoff can be achieved between the expected value of error variance and sensitivity to parameter variations. If the value of this design parameter is equivalent to 1, our method becomes the minisum estimator, and as it becomes large, our method approaches the minimax method. The freedom to optimize the estimator using this design parameter gives the designer the option of avoiding the two extreme cases represented by the minisum and minimax methods and improving, it is hoped, the performance of his robust estimator. This method also can be used to design estimators that are robust to various aircraft types of flight conditions.

The same design algorithm developed here also can be used to find the estimator gains that would make the estimator insensitive to the variation of the maneuver intensity. In addition, the same design concept employed in this Note could be used to design a class of robust controllers that would have the same properties stated here, i.e., the capability of trading-off between the expected value of the cost function and the sensitivity to parameter variations using the new design parameter.

References

[1]Haddad, W. M., and Bernstein, D. S., "Robust, Reduced-Order, Nonstrictly Proper State Estimation via the Optimal Projection Equations with Guaranteed Cost Bounds," *IEEE Transactions on Automatic Control,* Vol. AC-33, June 1988, pp. 591–595.

[2]Haddad, W. M., and Bernstein, D. S., "Steady-State Kalman Filtering with an H_∞ Error Bound," *Systems & Control Letters,* Vol. 12, 1989, pp. 9–16.

[3]Khargonekar, P. P., and Nagpal, K. N., "Filtering and Smoothing in an H_∞ Setting," *Proceedings of 28th IEEE Conference on Decision and Control* (Tampa, FL), 1989, pp. 415–420.

[4]Fitzgerald, R. J., "Simple Tracking Filters: Closed Form Solutions," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AES-17, No. 6, 1981, pp. 781–785.

[5]Ly, U., and Cannon, R. H., Jr., "A Direct Method for Designing Robust Optimal Control Systems," *Proceedings of the AIAA Guidance and Control Conference* (Palo Alto, CA), AIAA, New York, Aug. 1978, pp. 440–448.

[6]Vinkler, A., and Wood, L. J., "A Comparison of Several Techniques for Designing Controllers of Uncertain Dynamic Systems," *Proceedings of the IEEE Conference on Decision and Control* (San Diego), Inst. of Electrical and Electronics Engineers, Piscataway, NJ, 1978, pp. 31–38.

[7]Salmon, D. M., "Minimax Controller Design," *IEEE Transactions on Automatic Control,* Vol. AC-13, No. 4, 1968, pp. 369–376.

[8]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems,* Wiley-Interscience, New York, 1972.

[9]Wagie, D. A., and Skelton, R. E., "Model Reduction and Controller Synthesis in the Presence of Parameter Uncertainty," *Automatica,* Vol. 22, No. 3, 1986, pp. 295–308.

[10]Choi, S. S., and Sirisena, H. R., "Computation of Optimal Output Feedback Gains for Linear Multivariable Systems," *IEEE Transactions on Automatic Control,* Vol. AC-19, June 1974, pp. 257–258.

[11]Smetana, F. O., *Computer Assisted Analysis of Aircraft Performance Stability and Control,* McGraw-Hill, New York, 1984.

Stabilizability of Linear Quadratic State Feedback for Uncertain Systems

Rong-Jyue Wang* and Wen-June Wang†
National Central University, Chung Li, 320, Taiwan, Republic of China

Introduction

In recent years, the problem of designing a stabilizing feedback control for a linear system containing time-varying uncertainties has received considerable attention. Based on Lyapunov's direct method, Cheres et al.[1] introduced a nonlinear controller to stabilize uncertain systems under the assumption that the system satisfies the so-called "matching conditions." In Thorp and Barmish,[2] linear controllers were derived to deal with the same problem. Recently, Tsay et al.[3] applied the conventional linear quadratic optimal state feedback method to find the robust regulator for linear uncertain systems with matching conditions. Moreover, Schmitendorf[4] used the Riccati equation approach to the design of a stabilizing controller for a class of uncertain linear systems without a matching condition.

In this Note, we consider the same problem but the uncertainties, which may exist on the system matrix and/or input matrix, are decomposed into the matching and mismatching

Received Feb. 25, 1992; revision received July 2, 1992; accepted for publication July 13, 1992. Copyright © 1992 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

*Graduate Student, Department of Electrical Engineering.
†Professor, Department of Electrical Engineering.

portions. Because of the decomposition of the uncertainties, the system may possibly tolerate larger uncertainties. The considered controller gain is in "linear quadratic state feedback" form, i.e., $K = \rho B^T P$. This Note mainly establishes the relationships between the constant ρ and the solution of the Riccati equations such that the system is robustly stabilized.

System Description and Problem Formulation

Consider a linear uncertain dynamic system like

$$\dot{x}(t) = \{A + \Delta A[\omega(t)]\}x(t) + \{B + \Delta B[\omega(t)]\}u(t) \qquad (1)$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$; A and B are the nominal system matrix and input matrix, respectively, of the appropriate dimensions. Uncertain matrices $\Delta A[\omega(t)]$ and $\Delta B[\omega(t)]$ depend continuously on the uncertainty vector $\omega(t)$, which is Lebesgue measureable and within an allowable bound set at $\Omega \in \Re^r$ for all $t \in [0,\infty)$. The following assumptions may be needed (Thorp and Barmish[2]) : (θ1) (A, B) is controllable; (θ2) $\Omega \in \Re^r$ is a compact set; (θ3) there are continuous mappings $D(\cdot): \Omega \to \Re^{m\times n}$ and $E(\cdot): \Omega \to \Re^{m\times m}$ such that the uncertain matrices can be separated into two parts, matching portions (ΔA_m and ΔB_m) and mismatching portions ($\Delta\tilde{A}$ and $\Delta\tilde{B}$), as follows:

$$\Delta A[\omega(t)] = \Delta A_m[\omega(t)] + \Delta\tilde{A}[\omega(t)]$$

where

$$\Delta A_m[\omega(t)] = BD[\omega(t)] \qquad (2a)$$

$$\Delta B[\omega(t)] = \Delta B_m[\omega(t)] + \Delta\tilde{B}[\omega(t)]$$

where

$$\Delta B_m[\omega(t)] = BE[\omega(t)] \qquad (2b)$$

$\forall \omega(t) \in \Omega$, $t \in [0,\infty)$.

The main objective of this Note is to design a linear state feedback

$$u(t) = -Kx(t) \qquad (3)$$

such that the system (1) for all $\omega(t) \in \Omega$ is stabilized. Moreover, $\|x\|$ and $\|A\|$ denote the Euclidean norm of vector x and matrix A, respectively. The $\lambda(A)$ and $\lambda_{\min}(A)$ are the eigenvalue and minimum eigenvalue of matrix A, respectively.

Robust Stabilizability of the Uncertain System

With the aid of the Lyapunov theorem, we have the following theorem.

Theorem 1

Suppose all assumptions [(θ1), (θ2), and (θ3)] hold and the state feedback gain matrix is $K = \rho B^T P$, where $P = P^T > 0$ is the solution of Eq. (4)

$$A^T P + PA + 2\eta P - PBB^T P + Q = 0 \qquad (4)$$

where $Q = Q^T > 0$ and $\eta > 0$. Then the system (1) with Eq. (3) is asymptotically stable, if there exist ρ, Q, and η satisfying the following conditions, respectively:

$$[\rho - 1 - (1/\xi)]I + \rho[E(\omega) + E^T(\omega)] > 0 \qquad (5a)$$

$$Q > \xi D^T(\omega)D(\omega) + \epsilon I \qquad (5b)$$

$$\eta > (1/2)\left[(1/\epsilon)\|\Delta\tilde{A}(\omega)\|^2 + \rho\|\Delta\tilde{B}(\omega)\|^2\right]\|P\| \qquad (5c)$$

where both ϵ and ξ are some positive constants.

Proof. It is given in Appendix A.

Remark 1. The assumption of a matching condition of the uncertainties is necessary and two controllers (i.e., K and $-\gamma B_0^T P$) are needed in Jabbari and Schmitendorf.[8] However, we use only one controller $-\rho B^T P$ and also consider the mismatching uncertainties in this Note.

With a little modification of the proof of Theorem 1, we have the following.

Corollary 1

For the same system and assumptions in Theorem 1, if Q, ρ, and η *satisfy* Eqs. (5b), (6a), and (6b), respectively, then the system (1) with Eq. (3) is asymptotically stable.

$$[\rho - 1 - (1/2\xi)]I + (1/2)\rho[E(\omega) + E^T(\omega)] > 0 \qquad (6a)$$

$$\eta > (1/2)\left[(1/\epsilon)\|\Delta\tilde{A}(\omega)\|^2 + \rho^2\|\Delta\tilde{B}(\omega)\|^2\right]\|P\| \qquad (6b)$$

Proof: See Appendix B.

Remark 2. Since P is obtained from Eq. (4), Eq. (6b) or (5c) is therefore not guaranteed for the mismatching part $\Delta\tilde{A}(\omega)$ and $\Delta\tilde{B}(\omega)$. Hence, it is necessary to find a decomposition such that $\|\Delta\tilde{A}(\omega)\|$ and $\|\Delta\tilde{B}(\omega)\|$ are as small as possible.

Next, we consider the case with the matching conditions (i.e., $\Delta\tilde{A} = \Delta\tilde{B} = 0$).

Corollary 2

Suppose assumptions (θ1) and (θ2) hold. Moreover, $\Delta\tilde{A} = \Delta\tilde{B} = 0$ and $K = \rho B^T P$, where P is the solution of Eq. (4) with $\eta = 0$. Then the system (1) with Eq. (3) $K = \rho B^T P$ is asymptotically stable, if there exist ρ and Q satisfying Eqs. (7a) and (7b), respectively,

$$[\rho - (1/2) - (1/2\xi)]I + (1/2)\rho\cdot[E(\omega) + E^T(\omega)] > 0 \qquad (7a)$$

$$Q > \xi D^T(\omega)D(\omega) \qquad (7b)$$

where ξ is a positive constant.

Proof. It is similar to the proof in Appendix B if we let $\Delta\tilde{A} = \Delta\tilde{B} = 0$.

Remark 3. Since ρ is positive, $\lambda_{\min}[E(\omega) + E(\omega)^T]$ must be larger than -2 such that Eq. (7a) holds. However, $\lambda_{\min}[E(\omega) + E(\omega)^T] > -1$ is needed in Tsay et al.[3] and Thorp and Barmish.[2] Furthermore, Jabbari and Schmitendorf[8] needed the constraint $\|E\| < 1$. It is seen that Corollary 2 can tolerate larger uncertainty ΔB_m than their results do.

If ΔA and ΔB are unstructured uncertainties, i.e., we only know the bounds of $\|\Delta A\|$ and $\|\Delta B\|$, then the decomposition (2) cannot be used. In this case, the following corollary is applicable.

Corollary 3

If assumptions (θ1) and (θ2) hold, the state feedback (3) $K = \rho B^T P$ can stabilize the system (1), if there exist Q and η satisfying Eqs. (8a) and (8b), respectively,

$$Q > \epsilon I > 0 \qquad (8a)$$

$$\eta > (1/2)\left[(1/\epsilon)\|\Delta A(\omega)\|^2 + \rho\|\Delta B(\omega)\|^2\right]\|P\| \qquad (8b)$$

where both ϵ and ρ are some positive constants and $P = P^T > 0$ satisfies Eq. (9).

$$A^T P + PA + 2\eta P - \rho PBB^T P + Q = 0 \qquad (9)$$

Proof. It is similar to the proof in Appendix A if we let $\Delta A_m = \Delta B_m = 0$.

Remark 4. The system without the matching uncertainties has been discussed in Schmitendorf[4] also. However, the considered uncertainties of this Note need not be of the "rank-1" type (Schmitendorf[4] and Petersen and Hollot[9]).

Example

Consider the linearized lateral dynamics of the L-1011 aircraft (described in Mcruer and Bates[6] and Sobel et al.[7])

$$\begin{bmatrix} \dot\phi \\ \dot r \\ \dot p \\ \dot\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 \\ 0 & 0.249 & -1 & -5.2 \\ 0.0386 & -0.996 & -0.003 & -0.117 \end{bmatrix} \begin{bmatrix} \phi \\ r \\ p \\ \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -0.744 & -0.032 \\ 0.337 & -1.12 \\ 0.02 & 0 \end{bmatrix} \begin{bmatrix} \zeta_r \\ \zeta_a \end{bmatrix}$$

where

$$x(t) = \begin{bmatrix} \phi \\ r \\ p \\ \beta \end{bmatrix} \begin{matrix} \text{band angle (rad)} \\ \text{yaw rate (rad s}^{-1}) \\ \text{roll rate (rad s}^{-1}) \\ \text{sideslip angle (rad)} \end{matrix}$$

$$u(t) = \begin{bmatrix} \zeta_r \\ \zeta_a \end{bmatrix} \begin{matrix} \text{rudder deflection} \\ \text{aileron deflection} \end{matrix}$$

The uncertainty matrices ΔA and ΔB appear in the following:

$$\Delta A(\omega) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.0369\omega_{a1} & -0.0032\omega_{a2} & 0.115\omega_{a3} \\ 0 & -0.0281\omega_{a4} & 0.125\omega_{a5} & -0.1892\omega_{a6} \\ 0 & 0 & 0.0001\omega_{a7} & 0.0146\omega_{a8} \end{bmatrix}$$

$$\Delta B(\omega) = \begin{bmatrix} 0 & 0 \\ 0.0861\omega_{b1} & 0.0022\omega_{b2} \\ -0.1524\omega_{b3} & 0.339\omega_{b4} \\ 0.01\omega_{b5} & 0 \end{bmatrix}$$

where $-1 \le \omega_{ai} \le 1$ and $-1 \le \omega_{bj} \le 1$ for all i and j. However by using Corollary 1, we can carry out the decomposition as Eq. (2), for instance,

$$D(\omega) = 0.001 \times \begin{bmatrix} 0 & d_1 & d_2 & d_3 \\ 0 & d_4 & d_5 & d_6 \end{bmatrix}, \quad E(\omega) = 0.001 \times \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix}$$

$$\Delta\tilde A(\omega) = 0.0001 \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 & a_6 \\ 0 & a_7 & a_8 & a_9 \end{bmatrix}$$

$$\Delta\tilde B(\omega) = 0.0001 \times \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix}$$

where $d_1 = -48\omega_{a1}$, $d_2 = 4.3\omega_{a2} - 4\omega_{a5}$, $d_3 = -153\omega_{a3} - 6\omega_{a6}$, $d_4 = -15\omega_{a1} + 25\omega_{a4}$, $d_5 = 108\omega_{a5}$, $d_6 = -45\omega_{a3} + 167\omega_{a6}$, $e_1 = -115\omega_{b1} - 6\omega_{b3}$, $e_2 = -3\omega_{b2} + 13\omega_{b4}$, $e_3 = -35\omega_{b1} + 135\omega_{b3}$, $e_4 = -1.6\omega_{b2} - 298\omega_{b4}$, $a_1 = 7\omega_{a1} + 8\omega_{a4}$, $a_2 = 5\omega_{a5}$, $a_3 = -3\omega_{a3} + 9\omega_{a6}$, $a_4 = -6\omega_{a1} - \omega_{a4}$, $a_5 = -14_{a2} + 3\omega_{a5}$, $a_6 = 12\omega_{a3} - \omega_{a6}$, $a_7 = 10\omega_{a1}$, $a_8 = -\omega_{a2} + 2\omega_{a5} + \omega_{a7}$, $a_9 = 31\omega_{a3} + \omega_{a6} + 146\omega_8$, $b_1 = -6\omega_{b1} - \omega_{b3}$, $b_2 = -\omega_{b2} + \omega_{b4}$, $b_3 = 4\omega_{b1} + 8\omega_{b3}$, $b_4 = 8\omega_{b2} + 9\omega_{b4}$, $b_5 = 23\omega_{b1} + \omega_{b3} + 100\omega_5$, $b_6 = \omega_{b1} - 3\omega_{b3}$. Letting $\epsilon = 1$ and $\xi = 5$ and by Eqs. (6a), (6b), and (5b), we have $\rho = 2.152$ and $Q = \mathrm{diag}[1, 1.0946, 1.351, 2.6008]$

$$K = \rho B^T P = \begin{bmatrix} -0.7066 & -6.6361 & -0.8803 & 6.0652 \\ -2.8664 & -2.7120 & -2.8056 & 6.3023 \end{bmatrix}$$

and

$$\eta = 0.2$$

Hence, $u(t) = -Kx(t)$ is a stabilizing controller.

Conclusion

This Note has investigated the stabilizability of the uncertain systems when the uncertainties are decomposed into the matching and mismatching portions. Even though there have been many relative investigations of this problem, this Note has given alternative results, and because of the decomposition of the uncertainties, the tolerable uncertainties may be larger. Moreover, these results are very well applicable to the L-1011 aircraft system. It should be mentioned that the mismatching uncertainties cannot always be too large to satisfy Eq. (5c) [Eq. (6b) or (8b)]. This problem is still worth being studied in the future.

Appendix A: Proof of Theorem 1

First, a useful lemma is given.

Lemma A.[5] For any matrix X and Y with appropriate dimensions, we have

$$X^TY + Y^TX \le \epsilon X^TX + \frac{1}{\epsilon} Y^TY$$

for any constant $\epsilon > 0$.

Now, we start to prove Theorem 1. Considering the system (1), we construct a Lyapunov function $V(x) = x^TPx$, where $P = P^T > 0$ satisfies Eq. (4). Then the time derivative of $V(x)$ along the trajectory of the system (1) with state feedback (3) is

$$\begin{aligned} \dot V(x) &= x^TA^TPx + x^TPAx + 2x^TP\Delta A_m x + 2x^TP\Delta\tilde A x \\ &\quad - 2x^TPBKx - x^TP\Delta B_m Kx - x^TK^T\Delta B_m^T Px \\ &\quad - 2x^TP\Delta\tilde B Kx \end{aligned} \tag{A1}$$

Using Lemma A and the Riccati equation (4), we have

$$\begin{aligned} \dot V(x) &\le -x^TQx - (2\rho - 1)x^TPBB^TPx \\ &\quad - \rho x^TPB(E^T + E)B^TPx + (1/\xi)x^TPBB^TPx \\ &\quad + \xi x^TD^TDx + (1/\epsilon)x^TP\Delta\tilde A\Delta\tilde A^TPx + x^T\epsilon Ix \\ &\quad + \rho x^TPBB^TPx + \rho x^TP\Delta\tilde B\Delta\tilde B^TPx - 2\eta x^TPx \end{aligned} \tag{A2}$$

$$\begin{aligned} &\le -x^T(Q - \xi D^TD - \epsilon I)x - [\rho - 1 - (1/\xi)]\|B^TPx\|^2 \\ &\quad - \rho x^TPB(E^T + E)B^TPx + (1/\epsilon)\|\Delta\tilde A\|^2\|Px\|^2 \\ &\quad + \rho\|\Delta\tilde B\|^2\|Px\|^2 - 2\eta x^TPx \end{aligned} \tag{A3}$$

If Eqs. (5a), (5b), and (5c) hold, $\dot V(x) < 0$. The proof is completed. □

Appendix B: Proof of Corollary 1

The last term on the right-hand side of Eq. (A1) of Appendix A is decomposed into $x^TPBB^TPx + \rho^2x^TP\Delta\tilde B\Delta\tilde B^TPx$. Then we follow very much the same approach of Appendix A.

Acknowledgment

The authors would like to thank the National Science Council of the Republic of China for financial support under contract NSC 81-0404-E008-02.

References

[1]Cheres, E., Gutman, S., and Palmor, Z. J., "Stabilization of Uncertain Dynamic Systems Including State Delay," *IEEE Transactions on Automatic Control,* Vol. 34, No. 11, 1989, pp. 1199–1203.

[2]Thorp, J. S., and Barmish, B. R., "On Quarantee Stability of Ultimate Linear Systems via Linear Control," *Journal of Optimization Theory and Applications,* Vol. 35, No. 4, 1981, pp. 559–579.

[3]Tsay, S. C., Fong, I. K., and Kuo, T. S., "Robust Linear Quadratic Optimal Control for Systems with Linear Uncertainties," *International Journal of Control,* Vol. 53, No. 1, 1991, pp. 81–96.

[4]Schmitendorf, W. E., "Designing Stabilizing Controllers for Uncertain Systems Using the Riccati Equation Approach," *IEEE Transactions on Automatic Control,* Vol. 33, No. 4, 1988, pp. 376–379.

[5]Zhou, K., and Khargonekar, P. P., "Robust Stabilization of Linear Systems with Norm-Bounded Time-Varying Uncertainty," *Systems and Control Letters,* Vol. 10, No. 1, 1988, pp. 17–20.

[6]Mcruer, D. T., and Bates, C. L., "Dynamics of the Airframe," Navy Bureau Aeronaut. Rept. AE-61-411, 1952.

[7]Sobel, K. M., Banda, S. S., and Yeh, H. H., "Robust Control for Linear Systems with Structured State Space Uncertainty," *International Journal of Control,* Vol. 50, No. 5, 1989, pp. 1991–2004.

[8]Jabbari, F., and Schmitendorf, W. E., "A Noniterative Method for the Design of Linear Robust Controllers," *IEEE Transactions on Automatic Control,* Vol. 35, No. 8, 1990, pp. 954–957.

[9]Petersen, I. R., and Hollot, C. V., "A Riccati Equation Approach to the Stabilization of Uncertain Linear Systems," *Automatica,* Vol. 22, No. 4, 1986, pp. 397–411.

Minimum-Effort Interception of Multiple Targets

Joseph Z. Ben-Asher*
Tel-Aviv University, Tel-Aviv 69978, Israel

I. Introduction

A WELL-KNOWN fact about the widely used proportional navigation law is that it is the solution to the minimum-effort interception problem where a zero miss distance is required as the terminal boundary condition.[1] This result provides the motivation for studying this minimum-effort problem with more than one target. Norbutas[2] formulated a set of n two-point boundary-value problems coupled through their boundary conditions. For the general case, he suggested some numerical solutions for open-loop control. He succeeded in finding an analytical solution in feedback form (i.e., closed-loop) only for the two-target case. The main contribution of this work is a generalization of the closed-loop analytical solution to the n-target case. Both proportional navigation (i.e., the solution to the one-target case) and Norbutas' two-target optimal solution are shown to be special cases of the general feedback solution.

II. Problem Formulation

Assume n stationary targets located in the plane in the neighborhood of a nominal line of sight (LOS) at fixed normal displacements $x_1, x_2, \ldots, x_n$. Without any loss of generality, we let $x_n = 0$. The nominal times at which the interceptor approaches these targets (i.e., passes through their projections on the LOS) are $t_1, t_2, \ldots, t_n$.

Received March 15, 1992; revision received June 30, 1992; accepted for publication July 8, 1992. Copyright © 1992 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved.

*Adjunct Professor, Department of Electronic Systems. Member AIAA.

Under the assumption that the speed V is constant, the equations of motion of the interceptor (in the plane) are the following:

$$r'(t) = V\cos[\alpha(t)]$$
$$x'(t) = V\sin[\alpha(t)] \tag{1}$$

where t is the elapsed time, $r(t)$ the projection of the trajectory on the nominal LOS, $x(t)$ the normal displacement, $\alpha(t)$ the flight-path angle, and $()'$ signifies differentiation with respect to time. If $\alpha(t)$ is small, we get

$$r'(t) \approx V$$
$$x'(t) = V\alpha(t) \equiv u(t) \tag{2}$$

Assuming that the interceptor has direct control over its normal acceleration $a(t)$, we obtain the following state equations:

$$x'(t) = u(t)$$
$$u'(t) = a(t) \tag{3}$$

The optimal control problem is to minimize the control effort

$$J = \frac{1}{2}\int_0^{t_n} a^2(t)\,dt \tag{4}$$

subject to the following conditions:

$$x(t_1) = x_1$$
$$x(t_2) = x_2$$
$$\vdots$$
$$x(t_n) = x_n \tag{5}$$

and the initial conditions

$$x(t_0) = x_0$$
$$u(t_0) = u_0 \tag{6}$$

III. Problem Analysis

Let $H(x, u, a, p_x, p_u)$ be the usual Hamiltonian defined by

$$H(x, u, a, p_x, p_u) = \tfrac{1}{2}a^2 + p_x u + p_u a \tag{7}$$

The Euler-Lagrange's equations are

$$p_x'(t) = 0$$
$$p_u'(t) = -p_x(t) \tag{8}$$

By the minimum principle, $a(t)$ should minimize H, thus

$$a(t) = -p_u(t) \tag{9}$$

The transversality conditions are the following[1,2]:

$$p_x(t_i^+) = p_x(t_i^-) + \alpha_i \qquad i = 1, \ldots, n-1$$
$$p_u(t_i^+) = p_u(t_i^-) \qquad i = 1, \ldots, n-1 \tag{10}$$

and at the terminal time

$$p_u(t_n) = 0 \tag{11}$$